FRACTIONAL LAPLACIAN PHASE TRANSITIONS AND BOUNDARY REACTIONS: A GEOMETRIC INEQUALITY AND A SYMMETRY RESULT

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Abstract. We deal with symmetry properties for solutions of nonlocal equations of the type

$$(-\Delta)^s v = f(v) \qquad \text{in } \mathbb{R}^n,$$

where $s \in (0,1)$ and the operator $(-\Delta)^s$ is the so-called fractional Laplacian.

The study of this nonlocal equation is made via a careful analysis of the following degenerate elliptic equation

$$\begin{cases} -\mathrm{div}\,(x^\alpha \nabla u) = 0 & \text{on } \mathbb{R}^n \times (0,+\infty) \\ -x^\alpha u_x = f(u) & \text{on } \mathbb{R}^n \times \{0\} \end{cases}$$

where $\alpha \in (-1,1)$.

This equation is related to the fractional Laplacian since the Dirichlet-to-Neumann operator $\Gamma_\alpha : u|_{\partial\mathbb{R}^{n+1}_+} \mapsto -x^\alpha u_x|_{\partial\mathbb{R}^{n+1}_+}$ is $(-\Delta)^{\frac{1-\alpha}{2}}$.

More generally, we study the so-called boundary reaction equations given by

$$\begin{cases} -\mathrm{div}\,(\mu(x)\nabla u) + g(x,u) = 0 & \text{on } \mathbb{R}^n \times (0,+\infty) \\ -\mu(x) u_x = f(u) & \text{on } \mathbb{R}^n \times \{0\} \end{cases}$$

under some natural assumptions on the diffusion coefficient μ and on the nonlinearities f and g.

We prove a geometric formula of Poincaré-type for stable solutions, from which we derive a symmetry result in the spirit of a conjecture of De Giorgi.

Contents

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Introduction

This paper is devoted to some geometric results on the following equation

$$(-\Delta)^s v = f(v) \qquad \text{in } \mathbb{R}^n. \tag{1}$$

The operator $(-\Delta)^s$ is the fractional Laplacian and it is a pseudo-differential operator with symbol $|\eta|^{2s}$, with $s \in (0,1)$ – here, η denotes the variable in the frequency space. This operator, which is a nonlocal operator, can also be defined, up to a multiplicative constant, by the formula

$$(-\Delta)^s v(x) = P.V. \int_{\mathbb{R}^n} \frac{v(x) - v(y)}{|x-y|^{n+2s}}\, dy, \tag{2}$$

where $P.V.$ stands for the Cauchy principal value (see [Lan72] for further details).

Seen as an operator acting on distributional spaces, the quantity $(-\Delta)^s v$ is well-defined as long as v belongs to the space

$$\mathcal{L}_s = \left\{ v \in \mathcal{S}'(\mathbb{R}^n),\ \int_{\mathbb{R}^n} \frac{|v(x)|}{(1+|x|)^{n+2s}}\, dx < \infty \right\} \bigcap C^2_{\rm loc}(\mathbb{R}^n).$$

Notice in particular that smooth bounded functions are admissible for the fractional Laplacian. The L^1 assumption allows to make the integral in (2) convergent at infinity, whereas the additional assumption of $C^2_{\rm loc}$-regularity is here to make sense to the principal value[1] near the singularity.

From a probabilistic point of view, the fractional Laplacian is the infinitesimal generator of a Levy process (see, e.g., [Ber96]).

[1]For $v \in C^2_{\rm loc}(\mathbb{R}^n)$, the singular integral in (2) makes sense for any $s \in (0,1)$. Of course, it is possible to weaken such assumption depending on the values of s.

This type of diffusion operators arise in several areas such as optimization [DL76], flame propagation [CRS07] and finance [CT04]. Phase transitions driven by fractional Laplacian-type boundary effects have also been considered in [ABS98, CG08] in the Gamma convergence framework. Power-like nonlinearities for boundary reactions have also been studied in [CCFS98]

In this paper, we focus on an analogue of the De Giorgi conjecture [DG79] for equations of the type (1), namely, whether or not "typical" solutions possess one-dimensional symmetry.

One of the main difficulty of the analysis of this operator is its nonlocal character. However, it is a well-known fact in harmonic analysis that the power $1/2$ of the Laplacian is the boundary operator of harmonic functions in the half-space.

In [CS07b], the equivalence between (1) and the α-harmonic extension in the half-space has recently been proved. More precisely, if one considers the boundary reaction problem

$$\begin{cases} \operatorname{div}(x^\alpha \nabla u) = 0 & \text{on } \mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0, +\infty) \\ -x^\alpha u_x = f(u) & \text{on } \mathbb{R}^n \times \{0\}, \end{cases} \tag{3}$$

it is proved in [CS07b] that, up to a normalizing factor, the Dirichlet-to-Neumann operator $\Gamma_\alpha : u|_{\partial \mathbb{R}^{n+1}_+} \mapsto -x^\alpha u_x|_{\partial \mathbb{R}^{n+1}_+}$ is precisely $(-\Delta)^{\frac{1-\alpha}{2}}$ and then that $u(0, y)$ is a solution of

$$(-\Delta)^\alpha u(0, y) = f(u(0, y)).$$

Note that the condition $\frac{1-\alpha}{2} = s \in (0, 1)$ in (1) reduces to $\alpha \in (-1, 1)$.

Qualitatively, the result of [CS07b] states that one can localize the fractional Laplacian by adding an additional variable. This argument plays, for instance, a crucial role in the proof of full regularity of the solutions of the quasigeostrophic model as given by [CV06] and in the free boundary analysis in [CSS08].

The operator $\operatorname{div}(x^\alpha \nabla)$ is elliptic degenerate. However, since $\alpha \in (-1, 1)$, the weight x^α is integrable at 0. This type of weights falls into the category of A_2-Muckenhoupt weights (see, for instance, [Muc72]), and an almost complete theory for these equations is available (see [FKS82, FJK82]). In particular, one can obtain Hölder regularity, Poincaré-Sobolev-type estimates, Harnack and boundary Harnack principles.

In the present paper, we want to give a geometric insight of the phase transitions for equation (1). Our goal is to give a geometric proof of the one-dimensional symmetry result for fractional boundary reactions

in dimension $n = 2$, inspired by De Giorgi conjecture and in the spirit of the proof of Bernstein Theorem given in [Giu84].

A similar De Giorgi-type result for boundary reaction in dimension $n = 2$ has been proven in [CSM05] for $\alpha = 0$, which corresponds to the square root of the Laplacian in (1). The technique of [CSM05] will be adapted to the case $\alpha \neq 0$ in the forthcoming [CS07a].

However, the proofs in [CSM05, CS07a] are based on different methods (namely, a Liouville-type result inspired by [BCN97, AC00, AAC01] and a careful analysis of the linearized equation around a solution) and they are quite technical and long. Our techniques also gives some geometric insight on more general types of boundary reactions (see equation (4) below).

There has been a large number of works devoted to the symmetry properties of semilinear equations for the standard Laplacian. In particular, De Giorgi conjecture on the flatness of level sets of standard phase transitions has been studied in low dimensions in [AAC01, AC00, BCN97, GG98, GG03]. The conjecture has also been settled in [Sav08] up to dimension 8 under an additional assumption on the profiles at infinity.

Here we give a proof of analogous symmetry properties for phase transitions driven by fractional Laplacian as in (1). Such proof will be rather simple and short, with minimal assumptions (even on the nonlinearity f which can be taken here to be just locally Lipschitz) and it reveals some geometric aspects of the equation.

Indeed, our proof, which is based on the recent work [FSV07], relies heavily on a Poincaré-type inequality which involves the geometry of the level sets of u.

Most of our paper will focus on the boundary reaction equation in (3) (and, in fact, on the more general framework of (4) below). We recall that (3) still exhibits nonlocal properties. For instance, for $\alpha = 0$, it has been proven in [CSM05] that layer solutions admits nonlocal Modica-type estimates. Furthermore, in virtue of [CS07b], these equations can be considered as models of a large variety of nonlocal operators. As a consequence, it is worth studying the largest possible class of boundary reaction equations. We will then focus on the following problem:

$$\begin{cases} -\mathrm{div}\,(\mu(x)\nabla u) + g(x,u) = 0 & \text{on } \mathbb{R}^{n+1}_+ \\ -\mu(x) u_x = f(u) & \text{on } \partial\mathbb{R}^{n+1}_+. \end{cases} \tag{4}$$

under the following structural assumptions (denoted (S)):

- The function μ is in $L^1((0,r))$, for any $r>0$. Also, μ is positive and bounded over all open sets compactly contained in $\mathbb{R}^{n+1}_+$, i.e. for all $K \Subset \mathbb{R}^{n+1}_+$, there exists μ_1, $\mu_2>0$, possibly depending on K, such that $\mu_1 \le \mu(x) \le \mu_2$, for any $x\in K$.
- The function μ in an A_2-Muckenhoupt weight, that is, there exists $\kappa>0$ such that

$$\int_a^b \mu(x)\,dx \int_a^b \frac{1}{\mu(x)}\,dx \;\le\; \kappa(b-a)^2 \tag{5}$$

  for any $b\ge a\ge 0$.
- The map $(0,+\infty)\ni x\mapsto g(x,0)$ belongs to $L^\infty((0,r))$ for any $r>0$. Also, for any $x>0$, the map $\mathbb{R}\ni u\mapsto g(x,u)$ is locally Lipschitz, and given any R, $M>0$ there exists $C>0$, possibly depending on R and M in such a way that

$$\sup_{\substack{0<x<R\\ |u|<M}} |g_u(x,u)| \le C. \tag{6}$$

- The function f is locally Lipschitz in $\mathbb{R}$.

In Section 4, using a Poisson kernel extension, the fractional equation in (1) will be reduced to the extension problem in (3), which is a particular case of (4).

In our setting, (4) may be understood in the weak sense, namely supposing that $u\in L^\infty_{\rm loc}(\overline{\mathbb{R}^{n+1}_+})$, with

$$\mu(x)|\nabla u|^2 \in L^1(B_R^+) \tag{7}$$

for any $R>0$, and that[2]

$$\int_{\mathbb{R}^{n+1}_+} \mu(x)\nabla u\cdot\nabla\xi + \int_{\mathbb{R}^{n+1}_+} g(x,u)\,\xi = \int_{\partial\mathbb{R}^{n+1}_+} f(u)\xi \tag{8}$$

for any $\xi : B_R^+\to\mathbb{R}$ which is bounded, locally Lipschitz in the interior of $\mathbb{R}^{n+1}_+$, which vanishes on $\mathbb{R}^{n+1}_+\setminus B_R$ and such that

$$\mu(x)|\nabla\xi|^2\in L^1(B_R^+). \tag{9}$$

As usual, we are using here the notation $B_R^+ := B_R\cap\mathbb{R}^{n+1}_+$.

In the sequel, we will assume that u is stable, meaning that we suppose that

$$\int_{B_R^+} \mu(x)|\nabla\xi|^2 + \int_{B_R^+} g_u(x,u)\xi^2 - \int_{\partial B_R^+} f'(u)\xi^2 \;\ge\; 0 \tag{10}$$

[2]Condition (7) is assumed here to make sense of (8). We will see in the forthcoming Lemma 5 that it is always uniformly fulfilled when u is bounded.

The structural assumptions on g may be easily checked when $g(x,u)$ has the product-like form of $g^{(1)}(x)g^{(2)}(u)$.

for any ξ as above.

The stability (sometimes also called semistability) condition in (10) appears naturally in the calculus of variations setting and it is usually related to minimization and monotonicity properties. In particular, (10) says that the (formal) second variation of the energy functional associated to the equation has a sign (see, e.g., [MP78, FCS80, AAC01] and Section 7 of [FSV07] for further details).

The main results we prove are a geometric formula, of Poincaré-type, given in Theorem 1, and a symmetry result, given in Theorem 2.

For our geometric result, we need to recall the following notation. Fixed $x > 0$ and $c \in \mathbb{R}$, we look at the level set

$$S := \{y \in \mathbb{R}^n \text{ s.t. } u(y, x) = c\}.$$

We will consider the regular points of S, that is, we define

$$L := \{y \in S \text{ s.t. } \nabla_y u(y, x) \neq 0\}.$$

Note that L depends on the $x \in (0, +\infty)$ that we fixed at the beginning, though we do not keep explicit track of this in the notation.

For any point $y \in L$, we let ∇_L to be the tangential gradient along L, that is, for any $y_o \in L$ and any $G : \mathbb{R}^n \to \mathbb{R}$ smooth in the vicinity of y_o, we set

$$\nabla_L G(y_o) := \nabla_y G(y_o) - \left(\nabla_y G(y_o) \cdot \frac{\nabla_y u(y_o, x)}{|\nabla_y u(y_o, x)|} \right) \frac{\nabla_y u(y_o, x)}{|\nabla_y u(y_o, x)|}.$$

Since L is a smooth manifold, in virtue of the Implicit Function Theorem (and of the standard elliptic regularity of u apart from the boundary of $\mathbb{R}^{n+1}_+$), we can define the principal curvatures on it, denoted by

$$\kappa_1(y, x), \dots, \kappa_{n-1}(y, x),$$

for any $y \in L$. We will then define the total curvature

$$\mathcal{K}(y, x) := \sqrt{\sum_{j=1}^{n-1} \big(\kappa_j(y, x)\big)^2}.$$

We also define

$$\mathcal{R}^{n+1}_+ := \{(y, x) \in \mathbb{R}^n \times (0, +\infty) \text{ s.t. } \nabla_y u(y, x) \neq 0\}.$$

With this notation, we can state our geometric formula:

Theorem 1. *Let u be C^2_{loc} in the interior of $\mathbb{R}^{n+1}_+$. Assume that u is a bounded and stable weak solution of (4) under assumptions (S).*

Assume furthermore that for all $r > 0$,

$$|\nabla_y u| \in L^\infty(\overline{B_r^+}). \tag{11}$$

Then, for any $R > 0$ *and any* $\phi : \mathbb{R}^{n+1} \to \mathbb{R}$ *which is Lipschitz and vanishes on* $\mathbb{R}^{n+1}_+ \setminus B_R$*, we have that*

$$\int_{\mathcal{R}^{n+1}_+} \mu(x)\, \phi^2 \left(\mathcal{K}^2 |\nabla_y u|^2 + \big|\nabla_L |\nabla_y u|\big|^2 \right) \le \int_{\mathbb{R}^{n+1}_+} \mu(x)\, |\nabla_y u|^2 |\nabla \phi|^2.$$

Assumption (11) is natural and it holds in particular in the important case $g := 0$, $\mu(x) = x^\alpha$ where $\alpha \in (-1, 1)$, as discussed in Lemmata 9 and 13 below. Interior elliptic regularity also ensures that u is smooth inside $\mathbb{R}^{n+1}_+$.

The result in Theorem 1 has been inspired by the work of [SZ98a, SZ98b], as developed in [Far02, FSV07]. In particular, [SZ98a, SZ98b] obtained a similar inequality for stable solutions of the standard Allen-Cahn equation, and symmetry results for possibly singular or degenerate models have been obtained in [Far02, FSV07]. Related geometric inequalities also played an important role in [CC06].

The advantage of the above formula is that one bounds tangential gradients and curvatures of level sets of stable solutions in terms of the gradient of the solution. That is, suitable geometric quantities of interest are controlled by an appropriate energy term.

On the other hand, since the geometric formula bounds a weighted L^2-norm of any test function ϕ by a weighted L^2-norm of its gradient, we may consider Theorem 1 as a weighted Poincaré inequality. Again, the advantage of such a formula is that the weights have a neat geometric interpretation.

The second result we present is a symmetry result in low dimension:

Theorem 2. *Let the assumptions of Theorem 1 hold and let* $n = 2$.

Suppose also that one of the following conditions (12) *or* (13) *hold, namely assume that either for any* $M > 0$

$$\text{the map } (0, +\infty) \ni x \mapsto \sup_{|u| \le M} |g(x, u)| \text{ is in } L^1((0, +\infty)) \tag{12}$$

or that

$$\inf_{\substack{x \in \mathbb{R}^n \\ u \in \mathbb{R}}} g(x, u)\, u \ge 0. \tag{13}$$

Suppose also that there exists $C > 0$ *in such a way that*

$$\int_0^R \mu(x)\, dx \le C R^2 \tag{14}$$

for any $R \ge 1$.

Then, there exist $\omega \in \mathrm{S}^1$ *and* $u_o : \mathbb{R} \times [0, +\infty) \to \mathbb{R}$ *such that*

$$u(y, x) = u_o(\omega \cdot y, x)$$

for any $(y, x) \in \mathbb{R}^3_+$.

Roughly speaking, Theorem 2 asserts that, for any $x > 0$, the function $\mathbb{R}^2 \ni y \mapsto u(y, x)$ depends only on one variable. Thus, as remarked at the beginning of this paper, Theorem 2 may be seen as the analogue of De Giorgi conjecture of [DG79] in dimension $n = 2$ for equation (1).

Of course, condition (14) is satisfied, for instance, for $\mu := x^\alpha$ and $\alpha \in (-1, 1)$ and (12) is fulfilled by $g := 0$, or, more generally, by $g := g^{(1)}(x)g^{(2)}(u)$, with $g^{(1)}$ summable over $\mathbb{R}^+$ and $g^{(2)}$ locally Lipschitz. Also, condition (13) is fulfilled by $g := u^{2\ell+1}$, with $\ell \in \mathbb{N}$.

We remark that when u is not bounded, the claim of Theorem 2 does not, in general, hold (a counterexample being $\mu := 1$, $f := 0$, $g := 0$ and $u(y_1, y_2, x) := y_1^2 - y_2^2$).

Theorem 11 below will also provide a result, slightly more general than Theorem 2, which will be valid for $n \geq 2$ and without conditions (12) or (13), under an additional energy assumption.

The pioneering work in [CSM05] is related to Theorem 2. Indeed, with different methods, [CSM05] proved a result analogous to our Theorem 2 under the additional assumptions that $\alpha = 0$ and $f \in C^{1,\beta}$ for some $\beta > 0$ (see, in particular, page 1681 and Theorem 1.5 in [CSM05]). The method of [CSM05] will be adapted to the case $\alpha \in (-1, 1)$ in the forthcoming paper [CS07a].

We finally state the symmetry result for equation (1):

Theorem 3. *Let* $v \in C^2_{\mathrm{loc}}(\mathbb{R}^n)$ *be a bounded solution of equation* (1)*, with* $n = 2$ *and* f *locally Lipschitz.*

Suppose that either

$$f := 0 \tag{15}$$

or that

$$\partial_{y_2} v > 0. \tag{16}$$

Then, there exist $\omega \in \mathrm{S}^1$ *and* $v_o : \mathbb{R} \to \mathbb{R}$ *such that*

$$v(y) = v_o(\omega \cdot y)$$

for any $y \in \mathbb{R}^2$.

The remaining part of the paper is devoted to the proofs of Theorems 1 , 2, 3. For this, some regularity theory for solutions of equation (4) will also be needed.

1. Regularity theory for equation (4)

This section is devoted to several results we need for the regularity theory of equation (4). We do not develop here a complete theory.

We recall that

$$\mu(x)\, u_x^2 \in L^1(B_R^+) \tag{17}$$

for any $R > 0$, due to (7).

1.1. **Regularity for equation** (4) **under assumption** (11).

We start with an elementary observation:

Lemma 4. *If $n = 2$ and (14) holds, then there exists $C > 0$ in such a way that*

$$\int_{B_{2R}^+ \setminus B_R^+} \mu(x) \le C R^4 \tag{18}$$

for any $R \ge 1$.

Proof. Using (14), we have that

$$\begin{aligned}
\int_{B_{2R}^+ \setminus B_R^+} \mu(x) &\le \int_0^{2R} \int_{B_{2R}} \mu(x)\, dy\, dx \\
&\le C_1 R^2 \int_0^{2R} \mu(x)\, dx \\
&\le C_2 R^4,
\end{aligned}$$

for suitable C_1, $C_2 > 0$. ■

Though not explicitly needed here, we would like to point out that the natural integrability condition in (7) holds uniformly for bounded solutions. A byproduct of this gives an energy estimate, which we will use in the proof of Theorem 2.

Lemma 5. *Let u be a bounded weak solution of (4) under assumptions (S).*

Then, for any $R > 0$ there exists C, possibly depending on R, in such a way that

$$\|\mu(x) |\nabla u|^2\|_{L^1(B_R^+)} \le C.$$

Moreover, if

- *$n = 2$,*
- *either (12) or (13) holds,*
- *(14) holds,*

then there exists $C_o > 0$ such that

$$\int_{B_R^+} \mu(x)\,|\nabla u|^2 \le C_o\,R^2 \tag{19}$$

for any $R \geq 1$.

Proof. The proof consists just in testing the weak formulation in (8) with $\xi := u\tau^2$ where τ is a cutoff function such that $0 \le \tau \in C_0^\infty(B_{2R})$, with $\tau = 1$ in B_R and $|\nabla\tau| \le 8/R$, with $R \ge 1$. Note that such a ξ is admissible, since (9) follows from (7).

One then gets from (8) that

$$\begin{aligned}\int_{\mathbb{R}^{n+1}_+} \mu(x)\left(|\nabla u|^2\tau^2 + 2\tau\nabla u\cdot\nabla\tau\right) + \int_{\mathbb{R}^{n+1}_+} g(x,u)u\tau^2 \\ = \int_{\mathbb{R}^n} f(u)u\tau^2.\end{aligned}$$

Thus, by Cauchy-Schwarz inequality,

$$\begin{aligned}\int_{\mathbb{R}^{n+1}_+} \mu(x)\,|\nabla u|^2\tau^2 \;&\le\; \frac{1}{2}\int_{\mathbb{R}^{n+1}_+} \mu(x)\,|\nabla u|^2\tau^2 \\ &\quad + C_*\Big(\int_{\mathbb{R}^{n+1}_+} \mu(x)|\nabla\tau|^2 + \int_{\mathbb{R}^n} |f(u)|\,|u|\,\tau^2\Big) \\ &\quad - \int_{\mathbb{R}^{n+1}_+} g(x,u)\,u\,\tau^2,\end{aligned}$$

for a suitable constant $C_* > 0$.

This, recalling (12), (13) and (18), plainly gives the desired result. ■

We now control further derivatives in y, thanks to the fact that the operator is independent of the variable y:

Lemma 6. *Let u be a bounded weak solution of (4) under conditions (S). Suppose that (11) holds. Then,*

$$\mu(x)|\nabla u_{y_j}|^2 \in L^1(B_R^+)$$

for every $R > 0$.

Proof. Given $|\eta| < 1$, $\eta \neq 0$, we consider the incremental quotient

$$u_\eta(y,x) := \frac{u(y_1,\dots,y_j+\eta,\dots,y_n,x) - u(y_1,\dots,y_j,\dots,y_n,x)}{\eta}.$$

Since f is locally Lipschitz,

$$[f(u)]_\eta \le C, \tag{20}$$

for some $C > 0$, due to (11).

Analogously, from (6) and (11), for any $R > 0$ there exists $C_R > 0$ such that

$$[g(x,u)]_\eta \le C_R \tag{21}$$

for any $x \in (0,R)$.

Let now ξ be as requested in (8). Then, (8) gives that

$$\begin{aligned} & \int_{\mathbb{R}^{n+1}_+} \left[\mu(x)\nabla u_\eta \cdot \nabla\xi + \big(g(x,u)\big)_\eta \xi\right] - \int_{\partial\mathbb{R}^{n+1}_+} \left[f(u)\right]_\eta \xi \\ = & -\int_{\mathbb{R}^{n+1}_+} \left[\mu(x)\nabla u \cdot \nabla\xi_{-\eta} + g(x,u)\,\xi_{-\eta}\right] + \int_{\partial\mathbb{R}^{n+1}_+} f(u)\xi_{-\eta} \\ = & \ 0. \end{aligned}$$

We now consider a smooth cutoff function τ such that $0 \le \tau \in C_0^\infty(B_{R+1})$, with $\tau = 1$ in B_R and $|\nabla\tau| \le 2$. Taking $\xi := u_\eta \tau^2$ in the above expression, one gets

$$\begin{aligned} & 2\int_{\mathbb{R}^{n+1}_+} \mu(x)\tau u_\eta \nabla u_\eta \cdot \nabla\tau \\ & \quad + \int_{\mathbb{R}^{n+1}_+} \mu(x)\tau^2|\nabla u_\eta|^2 + \int_{\mathbb{R}^{n+1}_+} \big(g(x,u)\big)_\eta u_\eta\,\tau^2 \\ & \quad = \int_{\partial\mathbb{R}^{n+1}_+} \big(f(u)\big)_\eta\, u_\eta\tau^2. \end{aligned} \tag{22}$$

We remark that the above choice of ξ is admissible, since (9) follows from (11) and (17).

Now, by Cauchy-Schwarz inequality, we have

$$\begin{aligned} & \int_{\mathbb{R}^{n+1}_+} \mu(x)\tau u_\eta \nabla u_\eta \cdot \nabla\tau \ge -\frac{\varepsilon}{2}\int_{\mathbb{R}^{n+1}_+} \mu(x)\tau^2|\nabla u_\eta|^2 \\ & -\frac{1}{2\varepsilon}\int_{\mathbb{R}^{n+1}_+} \mu(x)u_\eta^2|\nabla\tau|^2 \end{aligned}$$

for any $\varepsilon > 0$.

Therefore, by choosing ε suitably small, (22) reads

$$\begin{aligned} & \int_{\mathbb{R}^{n+1}_+} \mu(x)\tau^2|\nabla u_\eta|^2 \\ \le & \ C\Big[\int_{B^+_{R+1}} \mu(x)u_\eta^2 + \int_{B^+_{R+1}} \big|\big(g(x,u)\big)_\eta u_\eta\big| \\ & \quad + \int_{\{|y|\le R\}\times\{x=0\}} \big|\big(f(u)\big)_\eta u_\eta\big|\Big]. \end{aligned}$$

for some $C > 0$.

From (11), (20) and (21), we thus control

$$\int_{B_R^+} \mu(x)\tau^2 |\nabla u_\eta|^2$$

uniformly in η.

By sending $\eta \to 0$ and using Fatou Lemma, we obtain the desired claim. ■

Following is the regularity needed for some subsequent computations:

Lemma 7. *Let u be $C^2_{\rm loc}$ in the interior of $\mathbb{R}^{n+1}_+$. Suppose that u is a bounded weak solution of (4) under conditions (S) and that (11) holds.*

Then,

$$\begin{aligned} &\text{for almost any } x>0, \text{ the map } \mathbb{R}^n \ni y \mapsto \nabla u(y,x) \\ &\text{is in } W^{1,1}_{\rm loc}(\mathbb{R}^n,\mathbb{R}^n) \end{aligned} \tag{23}$$

and

$$\begin{aligned} &\text{the map } \mathbb{R}^{n+1}_+ \ni (y,x) \mapsto \mu(x) \textstyle\sum_{j=1}^n \big(|\nabla u_{y_j}|^2 + |u_{y_j}|^2\big) \\ &\text{is in } L^1(B_r^+), \text{ for any } r>0. \end{aligned} \tag{24}$$

What is more,

$$\begin{aligned} &\text{the map } \mathbb{R}^{n+1}_+ \ni (y,x) \mapsto \mu(x)\big(|\nabla|\nabla_y u||^2 + |\nabla_y u|^2\big) \\ &\text{is in } L^1(B_r^+), \text{ for any } r>0. \end{aligned} \tag{25}$$

for any $r > 0$.

Proof. Since u is $C^2_{\rm loc}$ in the interior of $\mathbb{R}^{n+1}_+$, for any $x \in (\epsilon, 1/\epsilon)$ and any $R > 0$

$$\int_{B_R} |\nabla u(y,x)| + \sum_{j=1}^n |\nabla u_{y_j}(y,x)|\, dy \le C$$

for a suitable $C > 0$, possibly depending on ϵ and R, which proves (23).

Exploiting Lemma 6, (11) and the local integrability of $\mu(x)$, one obtains (24).

To prove (25), we now perform the following standard approximation argument. Define $\Gamma = (\Gamma_1, \dots, \Gamma_n) := \nabla_y u$, and let r, $\rho > 0$ and $P \in \mathbb{R}^{n+1}_+$ be such that $B_{r+\rho}(P) \subset \mathbb{R}^{n+1}_+$. Fix also $i \in \{1, \dots, n+1\}$.

Then, for any $\epsilon > 0$,

$$\frac{\sum_{j=1}^n \Gamma_j\,\partial_i\Gamma_j}{\sqrt{\epsilon^2+\sum_{j=1}^n\Gamma_j^2}} \le \frac{2\,|\Gamma|\,|\partial_i\Gamma|}{\epsilon+|\Gamma|} \le 2|\partial_i\Gamma| \in L^1(B_r(P))$$

$$\lim_{\epsilon\to 0^+}\frac{\sum_{j=1}^n \Gamma_j\,\partial_i\Gamma_j}{\sqrt{\epsilon^2+\sum_{j=1}^n\Gamma_j^2}} = \chi_{\{\Gamma\neq 0\}}\frac{\sum_{j=1}^n \Gamma_j\,\partial_i\Gamma_j}{|\Gamma|}$$

$$\sqrt{\epsilon^2+\sum_{j=1}^n\Gamma_j^2} \le \epsilon+|\Gamma| \in L^1(B_r(P))$$

and $$\lim_{\epsilon\to 0^+}\sqrt{\epsilon^2+\sum_{j=1}^n\Gamma_j^2} = |\Gamma|,$$

thanks to (24). As standard, we denote by χ_A, here and in the sequel, the characteristic function of a set A.

Therefore, by Dominated Convergence Theorem,

$$\begin{aligned}
\int_{\mathbb{R}^{n+1}_+}\psi\chi_{\{\Gamma\neq 0\}}\frac{\sum_{j=1}^n \Gamma_j\,\partial_i\Gamma_j}{|\Gamma|} &= \lim_{\epsilon\to 0^+}\int_{\mathbb{R}^{n+1}_+}\psi\frac{\sum_{j=1}^n \Gamma_j\,\partial_i\Gamma_j}{\sqrt{\epsilon^2+\sum_{j=1}^n\Gamma_j^2}}\\
&= \lim_{\epsilon\to 0^+}\int_{\mathbb{R}^{n+1}_+}\psi\,\partial_i\left(\sqrt{\epsilon^2+\sum_{j=1}^n\Gamma_j^2}\right)\\
&= -\lim_{\epsilon\to 0^+}\int_{\mathbb{R}^{n+1}_+}(\partial_i\psi)\sqrt{\epsilon^2+\sum_{j=1}^n\Gamma_j^2}\\
&= -\int_{\mathbb{R}^{n+1}_+}(\partial_i\psi)|\Gamma|.
\end{aligned}$$

for any $\psi\in C_0^\infty(B_r(P))$.

Thus, since P, r and ρ can be arbitrarily chosen, we have that

$$\partial_i|\Gamma| = \chi_{\{\Gamma\neq 0\}}\frac{\sum_{j=1}^n \Gamma_j\partial_i\,\Gamma_j}{|\Gamma|}$$

weakly and almost everywhere in $\mathbb{R}^{n+1}_+$.

Accordingly,

$$\begin{aligned}|\nabla|\nabla_y u||^2 = |\nabla|\Gamma||^2 &= \sum_{i=1}^{n+1}(\partial_i|\Gamma|)^2 \\ \leq \sum_{i=1}^{n+1}\left(\frac{\sum_{j=1}^{n}\Gamma_j\,\partial_i\Gamma_j}{|\Gamma|}\right)^2 &\leq \sum_{i=1}^{n+1}|\partial_i\Gamma|^2 \\ = \sum_{i=1}^{n+1}\sum_{j=1}^{n}(\partial_i u_{y_j})^2 &= \sum_{j=1}^{n}|\nabla u_{y_j}|^2.\end{aligned}$$

Then, (24) implies (25). ∎

1.2. **Verification of assumption** (11)**.** In this section, we show that (11) is always satisfied in the important case $g := 0$, $\mu(x) := x^\alpha$, with $\alpha \in (-1, 1)$.

More precisely, we state the following result, the proof of which can be found in [CS07a]:

Lemma 8. *Let u be a bounded weak solution of* (3) *and assume that f is locally Lipschitz. Then there exists a constant $C > 0$ depending on R and $\beta \in (0,1)$ such that*

- *the function u is Hölder-continuous of exponent β and*

$$\|u\|_{C^\beta(\overline{B_R^+})} \leq C,$$

- *for all $j = 1, ..., n$, the function u_{y_j} is Hölder-continuous of exponent β and*

$$\|u_{y_j}\|_{C^\beta(\overline{B_R^+})} \leq C. \tag{26}$$

We can now prove the following gradient bound, which says that (11) holds for bounded solutions of equation (3):

Lemma 9. *Let u be a bounded weak solution of* (3) *and assume that f is locally Lipschitz.*

Then, there exists a constant $C > 0$ such that

$$\|\nabla_y u\|_{L^\infty(\overline{\mathbb{R}^{n+1}_+})} \leq C.$$

Proof. From (26), $\nabla_y u$ is bounded in, say, $\overline{\mathbb{R}^{n+1}_+} \cap \{0 \leq x \leq 3\}$.

Now, in $\overline{\mathbb{R}^{n+1}_+} \cap \{x > 3\}$, equation (3) is nondegenerate and therefore, the gradient bound follows from standard elliptic theory. ∎

2. Proof of Theorem 1

Besides few technicalities, the proof of Theorem 1 consists simply in plugging the right test function in stability condition (10) and in using the linearization of (4) to get rid of the unpleasant terms. Following are the rigorous details of the proof.

By (23), we have that

$$\int_{\mathbb{R}^{n+1}_+} \mu(x)\nabla u_{y_j}\cdot\Psi = \int_0^\infty \mu(x)\int_{\mathbb{R}^n} \nabla u_{y_j}\cdot\Psi\,dy\,dx = -\int_{\mathbb{R}^{n+1}_+} \mu(x)\nabla u\cdot\Psi_{y_j}$$

for any $j = 1,\dots,n$ and any $\Psi \in C^\infty(\mathbb{R}^{n+1}_+, \mathbb{R}^n)$ supported in B_R.

Thus, making use of (8), we conclude that

$$\begin{aligned}
&\int_{\mathbb{R}^{n+1}_+} \mu(x)\nabla u_{y_j}\cdot\nabla\psi \\
&\quad = -\int_{\mathbb{R}^{n+1}_+} \mu(x)\nabla u\cdot\nabla\psi_{y_j} \\
&\quad = -\int_{\partial\mathbb{R}^{n+1}_+} f(u)\psi_{y_j} + \int_{\mathbb{R}^{n+1}_+} g(x,u)\psi_{y_j} \\
&\quad = \int_{\partial\mathbb{R}^{n+1}_+} \big(f(u)\big)_{y_j}\psi - \int_{\mathbb{R}^{n+1}_+} g_u(x,u)u_{y_j}\psi \\
&\quad = \int_{\partial\mathbb{R}^{n+1}_+} f'(u)u_{y_j}\psi - \int_{\mathbb{R}^{n+1}_+} g_u(x,u)u_{y_j}\psi
\end{aligned} \tag{27}$$

for any $j = 1,\dots,n$ and any $\psi \in C^\infty(\mathbb{R}^{n+1}_+)$ supported in B_R.

A density argument (recall (5) and see, e.g., Lemma 3.4, Theorem 2.4 and (2.9) in [CPSC94]), via (24), implies that (27) holds for $\psi := u_{y_j}\phi^2$,

where ϕ is as in the statement of Theorem 1, therefore

$$
\begin{aligned}
&\int_{\partial B_R^+} f'(u)|\nabla_y u|^2\phi^2 \\
&\quad = \sum_{j=1}^n \int_{B_R^+} \mu(x)\nabla u_{y_j}\cdot\nabla(u_{y_j}\phi^2) + \sum_{j=1}^n \int_{B_R^+} g_u(x,u)u_{y_j}^2\phi^2 \\
&\quad = \sum_{j=1}^n \int_{B_R^+} \mu(x)\left(|\nabla u_{y_j}|^2\phi^2 + u_{y_j}\nabla u_{y_j}\cdot\nabla\phi^2\right) \\
&\qquad + \sum_{j=1}^n \int_{B_R^+} g_u(x,u)u_{y_j}^2\phi^2 \\
&\quad = \int_{B_R^+} \mu(x)\left(\sum_{j=1}^n |\nabla u_{y_j}|^2\phi^2 + \phi\nabla\phi\cdot\nabla|\nabla_y u|^2\right) \\
&\qquad + \int_{B_R^+} g_u(x,u)|\nabla_y u|^2\phi^2.
\end{aligned} \tag{28}
$$

Now, we make use of (10) by taking $\xi := |\nabla_y u|\phi$ (this choice was also performed in [SZ98a, SZ98b, Far02, FSV07]; note that (11) and (25) imply (9) and so they make it possible to use here such a test function). We thus obtain

$$
\begin{aligned}
0 \;\le\; & \int_{B_R^+} \mu(x)\Big(\big|\nabla|\nabla_y u|\big|^2\phi^2 + |\nabla_y u|^2|\nabla\phi|^2 \\
& +2|\nabla_y u|\phi\nabla\phi\cdot\nabla|\nabla_y u|\Big) \\
& + \int_{B_R^+} g_u(x,u)|\nabla_y u|^2\phi^2 - \int_{\partial B_R^+} f'(u)|\nabla_y u|^2\phi^2.
\end{aligned}
$$

This and (28) imply that

$$
\begin{aligned}
&\int_{\mathbb{R}^{n+1}_+} \mu(x)\phi^2\Big(\sum_{j=1}^{n-1}\big(\partial_x u_{y_j}\big)^2 - \big(\partial_x|\nabla_y u|\big)^2 \\
&+\sum_{j=1}^{n-1}|\nabla_y u_{y_j}|^2 - \big|\nabla_y|\nabla_y u|\big|^2\Big) \\
&+\int_{\mathbb{R}^{n+1}_+} \mu(x)\phi\nabla\phi\cdot\big(\nabla|\nabla_y u|^2 - 2|\nabla_y u|\nabla|\nabla_y u|\big) \\
&\quad = \int_{B_R^+} \mu(x)\phi^2\Big(\sum_{j=1}^{n-1}|\nabla u_{y_j}|^2 - \big|\nabla|\nabla_y u|\big|^2\Big) \\
&\quad + \int_{\mathbb{R}^{n+1}_+} \mu(x)\phi\nabla\phi\cdot\big(\nabla|\nabla_y u|^2 - 2|\nabla_y u|\nabla|\nabla_y u|\big) \\
&\qquad \le \int_{\mathbb{R}^{n+1}_+} \mu(x)|\nabla\phi|^2|\nabla_y u|^2.
\end{aligned}
\tag{29}
$$

Let now r, $\rho > 0$ and $P \in \mathbb{R}^{n+1}_+$ be such that $B_{r+\rho}(P) \subset \mathbb{R}^{n+1}_+$. We consider γ to be either $|\nabla_y u|$ or u_{y_j}. In force of (24) and (25), we see that γ is in $W^{1,2}(B_r(P))$, and so in $W^{1,1}_{\rm loc}(B_r(P))$.

Thus, by Stampacchia Theorem (see, e.g., Theorem 6.19 in [LL97]), $\nabla\gamma = 0$ for almost any $(y, x) \in B_r(P)$ such that $\gamma(y) = 0$.

Hence, since P, r and ρ can be chosen arbitrarily, we have that $\nabla|\nabla_y u| = 0 = \nabla u_{y_j}$ for almost every (y, x) such that $\nabla_y u(y, x) = 0$.

Accordingly, (29) may be written as

$$
\begin{aligned}
&\int_{\mathcal{R}^{n+1}_+} \mu(x)\phi^2\left(\sum_{j=1}^{n-1}\big(\partial_x u_{y_j}\big)^2 - \big(\partial_x|\nabla_y u|\big)^2\right) \\
&+\int_{\mathcal{R}^{n+1}_+} \mu(x)\phi^2\left(\sum_{j=1}^{n-1}|\nabla_y u_{y_j}|^2 - \big|\nabla_y|\nabla_y u|\big|^2\right) \\
\le\ &\int_{\mathbb{R}^{n+1}_+} \mu(x)|\nabla\phi|^2|\nabla_y u|^2.
\end{aligned}
$$

Therefore, by standard differential geometry formulas (see, for example, equation (2.10) in [FSV07]), we obtain

$$
\begin{aligned}
&\int_{\mathcal{R}_+^{n+1}} \mu(x)\phi^2 \left(\sum_{j=1}^{n-1} \left(\partial_x u_{y_j}\right)^2 - \left(\partial_x |\nabla_y u|\right)^2 \right) \\
&+ \int_{\mathcal{R}_+^{n+1}} \mu(x)\phi^2 \left(\mathcal{K}^2 |\nabla_y u|^2 + \big|\nabla_L |\nabla_y u|\big|^2 \right) \\
\le &\int_{\mathbb{R}_+^{n+1}} \mu(x) |\nabla\phi|^2 |\nabla_y u|^2 .
\end{aligned}
\tag{30}
$$

We now note that, on $\mathcal{R}_+^{n+1}$,

$$
\left(\partial_x |\nabla_y u|\right)^2 = \left| \frac{\nabla_y u \cdot \nabla_y u_x}{\nabla_y u} \right|^2 \le |\nabla_y u_x|^2 = \sum_{j=1}^{n-1} \left(\partial_x u_{y_j}\right)^2 .
$$

This and (30) complete the proof of Theorem 1. ∎

3. Proof of Theorem 2

The strategy for proving Theorem 2 is to test the geometric formula of Theorem 1 against an appropriate capacity-type function to make the left hand side vanish. This would give that the curvature of the level sets for fixed $x > 0$ vanishes and so that these level sets are flat, as desired (for this, the vanishing of the tangential gradient term is also useful to take care of the possible plateaus of u, where the level sets are not smooth manifold: see Section 2.4 in [FSV07] for further considerations).

Some preparation is needed for the proof of Theorem 2. Indeed, Theorem 2 will follow from the subsequent Theorem 11, which is valid for any dimension n and without the restriction in either (12) or (13).

We will use the notation $X := (y, x)$ for points in $\mathbb{R}^{n+1}$.

Given $\rho_1 \le \rho_2$, we also define

$$
\mathcal{A}_{\rho_1,\rho_2} := \{X \in \mathbb{R}_+^{n+1} \text{ s.t. } |X| \in [\rho_1, \rho_2]\}.
$$

Lemma 10. *Let $R > 0$ and $h : B_R^+ \to \mathbb{R}$ be a nonnegative measurable function.*

For any $\rho \in (0, R)$, let

$$
\eta(\rho) := \int_{B_\rho^+} h.
$$

Then,

$$
\int_{\mathcal{A}_{\sqrt{R},R}} \frac{h(X)}{|X|^2}\, dX \le 2 \int_{\sqrt{R}}^{R} t^{-3} \eta(t)\, dt + \frac{\eta(R)}{R^2}.
$$

Proof. By Fubini Theorem,

$$\begin{aligned}
&\int_{\mathcal{A}_{\sqrt{R},R}} \frac{h(X)}{2|X|^2}\,dX \\
=\ & \int_{\mathcal{A}_{\sqrt{R},R}} \int_{|X|}^{R} t^{-3} h(X)\,dt\,dX + \int_{\mathcal{A}_{\sqrt{R},R}} \frac{h(X)}{2R^2}\,dX \\
=\ & \int_{\sqrt{R}}^{R} \int_{\mathcal{A}_{\sqrt{R},t}} t^{-3} h(X)\,dX\,dt + \frac{1}{2R^2} \int_{\mathcal{A}_{\sqrt{R},R}} h(X)\,dX \\
\le\ & \int_{\sqrt{R}}^{R} \int_{B_t^+} t^{-3} h(X)\,dX\,dt + \frac{1}{2R^2} \int_{B_R^+} h(X)\,dX,
\end{aligned}$$

from which we obtain the desired result. ∎

Theorem 11. *Let u be as requested in Theorem 1. Assume furthermore that there exists $C_o \ge 1$ in such a way that*

$$\int_{B_R^+} \mu(x)|\nabla u|^2 \le C_o\, R^2 \tag{31}$$

for any $R \ge C_o$.

Then there exist $\omega \in \mathrm{S}^{n-1}$ and $u_o : \mathbb{R} \times (0, +\infty) \to \mathbb{R}$ such that

$$u(y, x) = u_o(\omega \cdot y, x)$$

for any $(y, x) \in \mathbb{R}^{n+1}_+$.

Proof. From Lemma 10 (applied here with $h(X) := \mu(x)|\nabla u(X)|^2$) and (31), we obtain

$$\int_{\mathcal{A}_{\sqrt{R},R}} \frac{\mu(x)|\nabla u(X)|^2}{|X|^2} \le C_1 \log R \tag{32}$$

for a suitable C_1, as long as R is large enough.

Now we define

$$\phi_R(X) := \begin{cases} \log R & \text{if } |X| \le \sqrt{R}, \\ 2\log\big(R/|X|\big) & \text{if } \sqrt{R} < |X| < R, \\ 0 & \text{if } |X| \ge R \end{cases}$$

and we observe that

$$|\nabla \phi_R| \le \frac{C_2\, \chi_{\mathcal{A}_{\sqrt{R},R}}}{|X|},$$

for a suitable $C_2 > 0$.

Thus, plugging ϕ_R inside the geometric inequality of Theorem 1, we obtain

$$(\log R)^2 \int_{B^+_{\sqrt{R}} \cap \mathcal{R}^{n+1}_+} \mu(x) \left(\mathcal{K}^2 |\nabla_y u|^2 + \big|\nabla_L |\nabla_y u|\big|^2 \right) \leq C_3 \int_{\mathcal{A}_{\sqrt{R},R}} \frac{\mu(x)|\nabla_y u|^2}{|X|^2}$$

for large R.

Dividing by $(\log R)^2$, employing (32) and taking R arbitrarily large, we see that

$$\mathcal{K}^2 |\nabla_y u|^2 + \big|\nabla_L |\nabla_y u|\big|^2$$

vanishes identically on $\mathcal{R}^{n+1}_+$, that is $\mathcal{K} = 0 = \big|\nabla_L |\nabla_y u|\big|$ on $\mathcal{R}^{n+1}_+$.

Then, the desired result follows by Lemma 2.11 of [FSV07] (applied to the function $y \mapsto u(y,x)$, for any fixed $x > 0$). ■

We now complete the proof of Theorem 2. We observe that, under the assumptions of Theorem 2, estimate (31) holds, thanks to (19). Consequently, the hypotheses of Theorem 2 imply the ones of Theorem 11, from which the claim in Theorem 2 follows. ■

4. Proof of Theorem 3

We use Theorem 2 to prove Theorem 3. For this, given a function v satisfying (1), we select an extension[3] u satisfying (3) by a suitable Poisson kernel, whose theory has been developed in [CS07b].

For this, we use the following result of [CS07b]:

Lemma 12. *The function*

$$P(y,x) = C_{n,\alpha} \frac{x^{1-\alpha}}{\left(x^2 + |y|^2\right)^{\frac{n+1-\alpha}{2}}}$$

is a solution of

$$\begin{cases} -\mathrm{div}\,(x^\alpha \nabla P) = 0 & \text{on } \mathbb{R}^n \times (0,+\infty) \\ P = \delta_0 & \text{on } \mathbb{R}^n \times \{0\}, \end{cases} \tag{33}$$

where $\alpha \in (-1,1)$ and $C_{n,\alpha}$ is a normalizing constant such that

$$\int_{R^n} P(y,x)\,dy = 1.$$

[3]The extension is not, in general, unique. For instance, both the functions $u := 0$ and $u := x^{1-\alpha}$ satisfy $\mathrm{div}\,(x^\alpha \nabla u) = 0$ in $\mathbb{R}^{n+1}_+$ with $u = 0$ on $\partial\mathbb{R}^{n+1}_+$.

We now come to the proof of Theorem 3. Let v be a bounded solution of (1) and consider the function

$$u(y,x) = \int_{\mathbb{R}^n} P(y-z,x)\, v(z)\, dz = \int_{\mathbb{R}^n} P(\xi,x)\, v(y-\xi)\, d\xi. \tag{34}$$

Note that since $P(x,.) \in L^1(\mathbb{R}^n)$ and $v \in L^\infty(\mathbb{R}^n)$ and by the embedding $L^1 * L^\infty \subset L^\infty$, we have that u is bounded in $\mathbb{R}^{n+1}_+$ if v is bounded in $\mathbb{R}^n$.

We now prove the following regularity result.

Lemma 13. *Let v be bounded and $C^2_{\rm loc}(\mathbb{R}^n)$. Let u be given by* (34)*. Then, for all $R>0$ there exists a constant $C>0$ such that*

$$\|x^\alpha u_x\|_{L^\infty(\overline{B_R^+})} \le C.$$

Proof. Since P has unit mass, we have the relation

$$u(y,x) - v(y) = C_{n,\alpha} \int_{\mathbb{R}^n} \frac{x^{1-\alpha}(v(y-\xi)-v(y))}{(x^2+|\xi|^2)^{(n+1-\alpha)/2}} d\xi.$$

Therefore,

$$\begin{aligned} x^\alpha u_x &= x^\alpha \partial_x (u(y,x)-v(y)) \\ &= C_{n,\alpha} \int_{\mathbb{R}^n} \frac{\left[(1-\alpha)|\xi|^2 - nx^2\right](v(y-\xi)-v(y))}{(x^2+|\xi|^2)^{(n+3-\alpha)/2}} d\xi \end{aligned}$$

This bounds the quantity $x^\alpha u_x$ by

$$\int_{\mathbb{R}^n} \frac{|v(y-\xi)-v(y)|}{(x^2+|\xi|^2)^{(n+1-\alpha)/2}} d\xi$$

which is controlled by

$$\begin{aligned} &\int_{\mathbb{R}^n} \frac{|v(y-\xi)-v(y)|}{|\xi|^{(n+1-\alpha)}} d\xi \\ &\qquad \le \int_{|\xi|\ge 1} \frac{2\|v\|_{L^\infty(\mathbb{R}^n)}}{|\xi|^{(n+1-\alpha)}} d\xi + \int_{|\xi|\le 1} \frac{\|\nabla v\|_{L^\infty(B_1(y))}}{|\xi|^{(n-\alpha)}} d\xi. \end{aligned}$$

The last two terms are summable and one gets the bound

$$\|x^\alpha u_x\|_{L^\infty(\overline{B_R^+})} \le C\big(\|v\|_{L^\infty(\mathbb{R}^n)} + \|\nabla v\|_{L^\infty(B_{R+1})}\big),$$

as desired. ■

We now complete the proof of Theorem 3 via the following argument. We take u as defined in (34) and we observe that (11) and (17) are satisfied, thanks to the local integrability of $x^{-\alpha}$ and Lemmata 9 and 13.

Also, u is stable, because of either (15) or (16).

Indeed, if (15) holds, then (10) is obvious since $f := 0 =: g$ in this case.

If, on the other hand, (16) holds, then $u_{y_2} = P * v_{y_2} > 0$ in $\mathbb{R}^{n+1}_+$, and $u_{y_2}(y,0) = v_{y_2}(y) > 0$ on $\partial\mathbb{R}^{n+1}_+$, thanks to Lemma 8.

Therefore, given $\xi : B^+_R \to \mathbb{R}$ which is bounded, locally Lipschitz in the interior of $\mathbb{R}^{n+1}_+$, which vanishes on $\mathbb{R}^{n+1}_+ \setminus B_R$ and such that (9) holds, we use (27) with $\psi := \xi^2/u_{y_2}$ (here, $j := 2$, $g := 0$, $\mu := x^\alpha$ and (24) make the choice of such a ψ admissible), and we get

$$\int_{\partial\mathbb{R}^{n+1}_+} f'(u)\xi^2 = \int_{\mathbb{R}^{n+1}_+} 2x^\alpha \xi \frac{\nabla u_{y_2} \cdot \nabla \xi}{u_{y_2}} - x^\alpha \xi^2 \frac{|\nabla u_{y_2}|^2}{u_{y_2}^2}.$$

This, by Cauchy-Schwarz inequality, gives (10) and so u is stable.

Then, we apply Theorem 2 to get that $u(y,x) = u_o(\omega \cdot y, x)$ for any $y \in \mathbb{R}^2$ and any $x > 0$, for an appropriate direction ω.

By Lemma 8, u is continuous up to $\{x = 0\}$ and so $u(y,0) = u_o(\omega \cdot y, 0)$.

Since, by (33) and (34),

$$u|_{\partial\mathbb{R}^{n+1}_+} = v,$$

the proof of Theorem 3 is complete. ■

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